

Networks and Markets — Lecture 3: Mechanism Design and Auctions

Nikhil Garg, Cornell Tech

1 Intro to auctions

Suppose we want to sell a single item to a group of bidders.

- Each bidder i submits a bid b_i .
- The auction needs to decide:
 - who gets the item (allocation rule),
 - how much they pay (payment rule).

1.1 Common single-item auction formats

First-price auction. Allocate the item to $\arg \max_i b_i$ and charge the winner their bid. For example, if bidder 1 bids 10, bidder 2 bids 15, and bidder 3 bids 12, then bidder 2 wins and pays 15.

Second-price auction. Allocate the item to $\arg \max_i b_i$ and charge the winner the *second-highest* bid. For example, if bidder 1 bids 10, bidder 2 bids 15, and bidder 3 bids 12, then bidder 2 wins and pays 12.

All-pay auction. Allocate the item to the highest bidder and charge *everyone* their bid. For example, if bidder 1 bids 10, bidder 2 bids 15, and bidder 3 bids 12, then bidder 2 wins and pays 15, bidder 1 pays 10, and bidder 3 pays 12. Only the winner gets the item.

1.2 Values and utilities

Each bidder i has a value for the item v_i .

- If bidder i gets the item and pays p_i , then their utility is $u_i = v_i - p_i$.
- If bidder i does not get the item and does not pay, then their utility is $u_i = 0$.
- In an all-pay auction, if bidder i does not get the item but pays p_i , then their utility is $u_i = -p_i$.

1.3 First-price auctions: bid shading

Suppose you value the item at v_i . What should you bid?

Of course, $0 \leq b_i \leq v_i$. Bidding above your value ($b_i > v_i$) risks winning and paying more than the item is worth to you, resulting in negative utility. Bidding exactly your value ($b_i = v_i$) guarantees zero utility if you win.

If you bid x , then your utility is

$$u_i(x) = \begin{cases} v_i - x & \text{if you win,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $G(x)$ denote the probability that you win if you bid x . Equivalently,

$$G(x) = \Pr(\text{second-highest bid} \leq x) = \Pr(\text{you win if you bid } x).$$

Then your expected utility from bidding x is

$$f(x) = (v_i - x)G(x) + 0 \cdot (1 - G(x)).$$

This highlights the basic tradeoff behind *bid shading*. The best bid is

$$x^* \in \arg \max_x f(x),$$

and typically $x^* < v_i$.

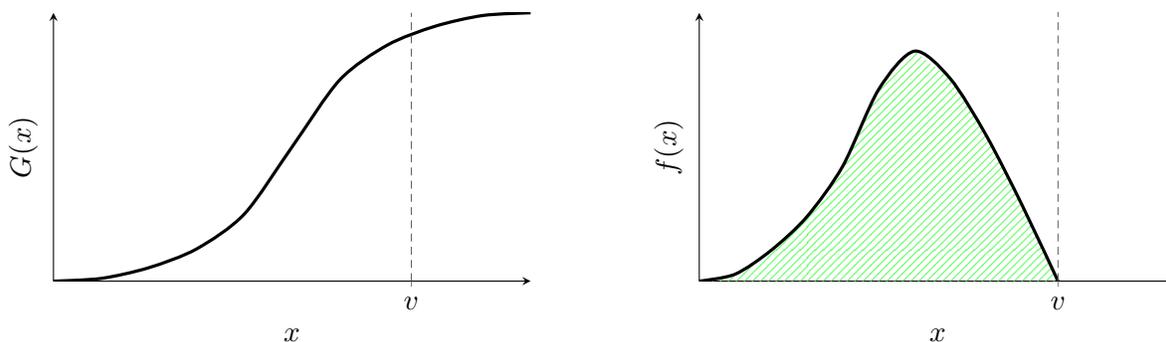


Figure 1: Illustration of $G(x)$ and $f(x) = (v_i - x)G(x)$ (shown with $v_i = 1$; dashed line marks v).

1.4 Second-price auctions

If you bid x , then your utility is

$$u_i(x) = \begin{cases} v_i - (\text{second-highest bid}) & \text{if you win,} \\ 0 & \text{if you lose.} \end{cases}$$

Claim (strategy-proofness). In a second-price auction, a (weakly) dominant strategy is to bid your value: $b_i = v_i$.

Proof sketch. Fix the other bids b_{-i} and let $B = \max b_{-i}$ denote the highest bid among the other bidders. (Assume no ties, for simplicity.)

- If you bid $b_i < B$, you lose and your utility is 0.
- If you bid $b_i > B$, you win and you pay B (the second-highest bid), so your utility is $v_i - B$.

Now compare any bid b_i to the truthful bid v_i .

- If $v_i < B$, then $v_i - B < 0$, so you prefer to lose (utility 0). Bidding $b_i = v_i$ makes you lose, whereas any $b_i > B$ would make you win and get negative utility.
- If $v_i > B$, then $v_i - B > 0$, so you prefer to win. Bidding $b_i = v_i$ makes you win and pay B . Any $b_i < B$ makes you lose and get 0, and any $b_i > v_i$ still makes you win and pay B (same utility).

Thus, bidding v_i is weakly dominant.

Claim (efficiency). If all bidders bid truthfully in a second-price auction, then the item is allocated to the bidder with the highest value.

Revenue equivalence principle (informal). Under standard assumptions, many common auction formats (e.g., first-price and second-price) generate the same expected revenue in equilibrium.

Why? Intuitively, while in a first-price auction bidders shade their bids down, the winner pays their bid; in a second-price auction bidders bid their values truthfully, but the winner pays the second-highest bid. These effects roughly cancel out.

1.5 Generalized second-price auctions (GSP) and VCG

When selling multiple “slots” (e.g., ad slots), a common extension is a *generalized second-price* auction. More generally, the *Vickrey–Clarke–Groves (VCG)* family of mechanisms achieves truthful reporting as a dominant strategy in a broad class of allocation problems.

1.6 Credibility (informal)

- First-price auctions can be viewed as “credible” in the sense that the rule does not rely on the seller reporting (or committing to) a second-highest bid.
- Second-price auctions are often described as “not credible” without an additional trust/commitment assumption.
- Question: does there exist an auction that is both strategy-proof and credible (in a static, one-shot setting)? No.

For more on credibility, see class website for references.