Networks and Markets

Homework 2

Due: March 3, 2026, 11:59pm

Submit solutions as a PDF file to Gradescope. On Gradescope, match pages with the corresponding problem (we will make a one-point deduction per problem if pages are not matched). Show work throughout, with legible handwriting. Clearly mark your answer by putting a box around it.

1 The Median Mechanism (12 pts)

Consider the following problem. Several people each have a preference that can be represented on the real number line. For example, three people might have preferences $-1, 4$, and 0.5. Our goal is to aggregate these preferences in a way to choose a single outcome x, such that a person with true preference p receives utility $-|x-p|$. So a person receives higher utility if the outcome x is closer to their true preference. (For example, suppose x is our choice for how many quizzes we should have in class, and each of you has different preferences for this number).

Now consider three agents, Euler, Fermat, and Gauss, who have true preferences $-1, 2$, and 7 respectively.

Part (a) (2 pts) Suppose $x = 3$. What are Euler, Fermat, and Gauss's three utilities? Answer as an ordered triple (Euler's utility, Fermat's utility, Gauss's utility).

Part (b) (3 pts) Define the social welfare as the sum of Euler's, Fermat's, and Gauss's utilities. Give one outcome x that maximizes the social welfare. Show your work.

Part (c) (3 pts) Suppose that we do not know the agents' true preferences, but rather have them report their preferences. We need a way to aggregate these reported preferences into an outcome x. Consider the *mean mechanism*, where we choose x to be the mean (average) of the three reported preferences. Show that reporting truthfully is not a Nash equilibrium. Hint: suppose that two agents report the truth. What does the third agent want to do?

Part (d) (4 pts) Instead consider the *median mechanism.* Show that reporting truthfully is a Nash equilibrium. *Hint:* In fact, one can show that reporting truthfully is a *dominant strategy*, meaning that the median mechanism is strategyproof: no matter what other agents do, each agent cannot benefit by lying.

2 Tie Breaking in School Matching (20 pts)

In this problem, we will consider a significant policy problem that arose in the design of school matching: the use of a single tie breaking (STB) or multiple tie breaking (MTB). Preferences of schools over students are often determined by lottery. The question was whether a single lottery number should be used to determine a student's priority at all schools, or whether a different lottery number should be used at each school.

Intuitively, many parents and students felt that STB was unfair: a single bad lottery number would mean that a student would have a low priority at *all* schools. As it turns out, however, STB actually matches more students to their top choices, and is generally considered better. Consequently, New York City and many other school districts use STB.

To analyze STB versus MTB, we introduce a model with a *continuum* of students. By considering an infinite number of students, it is actually easier to analyze deferred acceptance.

There is a continuum of students of total mass 1 and two schools each with capacity 1/3. Intuitively, this means that each school has capacity for a third of the total amount of students. Each student has a probability 1/2 of preferring school 1 and a probability 1/2 of preferring school 2.

Each student also has a random pair of lottery numbers (ℓ_1, ℓ_2), where $\ell_1, \ell_2 \in (0, 1)$. (We will specify this randomness later.) Now suppose schools 1 and 2 have **cutoffs** P_1 and P_2. Then we say that a student can **afford** school i if and only if $\ell_i > P_i$, i.e., if and only if their lottery number exceeds the school's cutoff. For example, if $P_1 = 0.5$ and $P_2 = 0.7$, then a student with lottery numbers $(0.3, 0.9)$ can afford school 2 but not school 1.

Given a choice of cutoffs, a student is matched to their most preferred school among the schools they can afford. For example, even if the student above prefers school 1, they are matched to school 2 since that is the only school they can afford.

Cutoffs P_1 and P_2 are **market-clearing** if and only if the probabilities a student is matched to school 1 and school 2 are each 1/3. Intuitively, this means that the right number of students is matched to each school. If cutoffs are market-clearing, then the corresponding matches exactly correspond to the outcome of deferred acceptance. *Side note: this is non-trivial to show.*

We will now walk through how to compute market-clearing cutoffs in one case. Consider a single student. A single lottery number is used at both schools, meaning that $(\ell_1, \ell_2) = (\ell, \ell)$ where ℓ is drawn uniformly at random from $(0, 1)$.

Part (a) (1 points). Consider cutoffs $P_1 = P_2 = 1/4$. What is the probability that the student can afford school 1? *Hint: In other words, what is the probability that the student's lottery number is higher than P_1?*

Part (b) (2 points). Consider cutoffs $P_1 = P_2 = 1/4$. What is the probability that the student matches with school 1? *Hint: What is the difference between affording a school versus matching to it?* Briefly explain.

Part (c) (1 points). Briefly explain why the cutoffs $P_1 = P_2 = 1/4$ are not market-clearing.

Part (d) (3 points). There exists P such that the cutoffs $P_1 = P_2 = P$ are market-clearing. What is P? Verify that the probabilities the student is matched with schools 1 and 2 are each 1/3.

Part (e) (2 points). Given the market-clearing cutoffs above, what is the probability that the student matches with their most preferred school?

Now consider the same setup, except now a different lottery number is used at each school for the student, meaning that ℓ_1 and ℓ_2 are each drawn independently and uniformly at random from $(0, 1)$.

Part (f) (1 points). Consider cutoffs $P_1 = P_2 = 1/4$. What is the probability that the student can afford school 1 but not school 2?

Part (g) (3 points). Consider cutoffs $P_1 = P_2 = 1/4$. What is the probability that the student matches with school 1? Briefly explain.

Part (h) (3 points). There exists P such that the cutoffs $P_1 = P_2 = P$ are market-clearing. What is P? Verify that the probabilities the student is matched with schools 1 and 2 are each $1/3$.

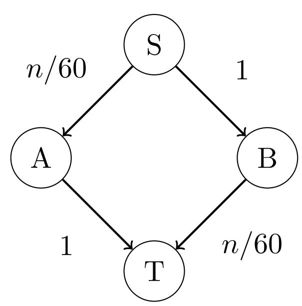
Part (i) (2 points). Given the market-clearing cutoffs above, what is the probability that the student matches with their most preferred school?

Part (j) (2 points). Compare your answers to (e) and (i). What does this comparison reveal about the choice between STB and MTB in school matching?

3 Braess's Paradox (15 pts)

In this problem, we explore a counterintuitive phenomenon in traffic networks: adding a new road can make traffic *worse* for everyone.

Consider a traffic network where 60 drivers commute from city S to city T each day. All 60 drivers depart at the same time. There are two intermediate cities, A and B, forming a diamond-shaped network:



The travel time on each road depends on the number of drivers n using that road:

- S→A: travel time = $n/60$ hours
- A→T: travel time = 1 hour (constant)
- S→B: travel time = 1 hour (constant)
- B→T: travel time = $n/60$ hours

At equilibrium, all drivers choose routes to minimize their own travel time, and no driver can reduce their travel time by switching routes.

Part (a) (5 pts) Initially only the four roads above exist. What are all possible equilibria? What is the total travel time of all drivers in each equilibrium?

Part (b) (5 pts) Now a new road A→B is added with travel time 0. What are all possible equilibria? What is the total travel time of all drivers in each equilibrium?

Part (c) (3 pts) The *Price of Anarchy* (PoA) measures the inefficiency of selfish behavior:

$$\text{PoA} = \frac{\text{Total cost at worst Nash equilibrium}}{\text{Minimum possible total cost}}.$$

After the new road is built, what is the social optimum (minimum total travel time if a central planner could assign routes)? What is the PoA?

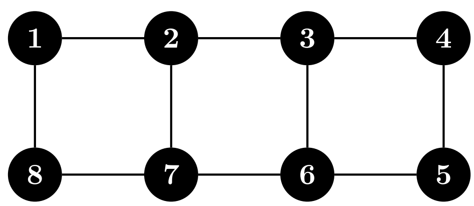
Part (d) (2 pts) Explain intuitively why adding the new road made the total travel time worse.

4 Network Effects (18 pts)

In this problem, we consider a simple model of network effects. The basic idea behind network effects is that for some products, the fact that many people use the product increases its value. For example, if many of your friends use WhatsApp to communicate, this directly increases the value of WhatsApp for you.

In our model, there is an underlying graph G of users. There are two products A and B. At time $t = 0$, everybody uses product B except for a set of *initial adopters*, who use product A. Then, at each subsequent time step, if at least a p proportion of a user's neighbors use product A, then that user will switch to product A. This continues until no further changes occur. Intuitively, a higher p means that a user will only want to switch if many of their neighbors adopt product A.

Now consider the following network of eight users, labeled 1 through 8.



Part (a) (3 points). Suppose that the set of initial adopters is $\{1, 2\}$ and $p = 0.2$. Which users end up adopting product A?

Part (b) (3 points). Suppose that the set of initial adopters is $\{3, 4\}$ and $p = 0.5$. Which users end up adopting product A?

Part (c) (3 points). Suppose that $p = 0.4$. Give a set of two initial adopters such that all users in the network end up adopting product A.

Part (d) (3 points). Suppose that $p = 2/3$. Give a set of two initial adopters such that no additional users adopt product A.

Part (e) (3 points). For which values of p is the following true: any choice of a single initial adopter will result in every user eventually adopting product A?

Part (f) (3 points). Suppose $p = 0.5$. What is the minimum number of initial adopters needed to guarantee that all users eventually adopt product A, regardless of which users are chosen as initial adopters?

5 The Stable Roommate Problem (18 pts)

In this problem, we explore matching markets where there are no "sides." Unlike school-student matching where schools and students form two distinct groups, here any agent can potentially be matched with any other agent.

Consider 4 people, Alice (A), Bob (B), Carol (C), and David (D), who need to be paired up into 2 pairs to share apartments. Each person has preferences over who they want to live with.

A matching is **stable** if there is no *blocking pair*: two people who would both prefer to live with each other rather than with their current roommates.

Part (a) (3 pts) Consider the following preferences (where A: B > C > D means A prefers B most, then C, then D):

- A: B > C > D
- B: A > C > D
- C: A > B > D
- D: A > B > C

Find a stable matching. Verify your answer by checking that no blocking pair exists.

Part (b) (4 pts) Now consider these preferences:

- A: B > C > D
- B: C > A > D
- C: A > B > D
- D: A > B > C

List all three possible matchings and determine which (if any) are stable.

We now consider the stable roommate problem for arbitrary preferences with n agents, where n is even.

Part (c) (6 pts) Prove that for any even $n \geq 4$, there exists a preference profile for n agents such that no stable matching exists.

Part (d) (5 pts) Prove or disprove: If a stable matching exists for a roommate problem with $n = 4$ agents, it must be unique.